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The Uniform Convergence of the Tau Method for Singularly Perturbed Problems

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Abstract—We prove the stability and uniform convergence of Tau Method approximations to the analytic solution of a model singularly perturbed boundary value problem, independently of the values taken by the singularity parameter ϵ.

Keywords—Singular perturbation, Differential equations, Tau Method.

1. INTRODUCTION

Let us consider a singularly perturbed boundary value model problem, widely discussed in recent literature through a variety of numerical techniques:

$$\begin{aligned} -\epsilon y''(x) + y(x) = 0, \qquad & x \in [-1,1], \\ y(-1) = \gamma_1, \quad y(1) = \gamma_2, \qquad & 0 < \epsilon \ll 1. \end{aligned} \tag{1}$$

Due to the existence of boundary layers, standard discretization techniques experience difficulties and require *ad hoc* procedures to give accurate approximations to the solution of problem (1) when ϵ is very close to zero.

In this note, we consider the approximation problem from the standpoint of the recursive formulation of the Tau Method [1] and show that, using that method, $y(x)$ can be approximated to any required accuracy, independently of ϵ. More precisely, we prove that:

(i) when ϵ tends to zero, the tau parameters and the approximation error have finite limits which are independent of ϵ, and
(ii) when the degree of approximation n tends to infinity, the Tau Method approximation converges uniformly to the analytic solution of (1).

The treatment of stiff and boundary-layer problems with the Tau Method has been discussed in [2–4]. In particular, the results given in this paper disprove a negative conjecture of Pfeifer and Roos [5] on the convergence of the Tau Method for singularly perturbed problems.

2. ASYMPTOTIC VALUES OF THE TAU PARAMETERS

For all $k \in \mathbf{N}$, the k^{th} canonical polynomial (see [1]) associated with the operator defining (1) is given by

$$Q_k(x) := k! \sum_{i=0}^{[k/2]} \frac{x^{k-2i}}{(k-2i)!} \epsilon^i. \tag{2}$$

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Let $n \in \mathbf{N}$ and let $V_n(x) := \sum_{k=0}^{n} c_k^n x^k$ stand for the n^{th} degree component of a Chebyshev or Legendre polynomial basis. To find a Tau Method approximation of order n, we solve

$$\begin{gathered} -\epsilon y_n''(x) + y_n(x) = (\tau_0 + \tau_1 x)\, V_n(x), \qquad x \in [-1,1], \\ y(-1) = \gamma_1, \qquad y(1) = \gamma_2, \end{gathered} \tag{3}$$

and get

$$y_n(x) = \tau_0 \left(\sum_{k=0}^{n} c_k^n Q_k(x) \right) + \tau_1 \left(\sum_{k=0}^{n} c_k^n Q_{k+1}(x) \right). \tag{4}$$

If n is, say, even, by (2)

$$\begin{aligned} \sum_{k=0}^{n} c_k^n Q_k(x) &= \sum_{p=0}^{n} c_{2p}^n Q_{2p}(x) = \sum_{p=0}^{n/2} c_{2p}^n (2p)! \sum_{i=0}^{p} \frac{x^{2p-2i}}{(2p-2i)!} \epsilon^i \\ &= \sum_{p=0}^{n/2} \left\{ \sum_{i=p}^{n/2} c_{2i}^n (2i)! \frac{x^{2i-2p}}{(2i-2p)!} \right\} \epsilon^p, \end{aligned}$$

and in particular, when $x = \pm 1$,

$$\sum_{k=0}^{n} c_k^n Q_k(1) = \sum_{k=0}^{n} c_k^n Q_k(-1) = \sum_{p=0}^{n/2} \lambda_{p,0} \epsilon^p, \tag{5}$$

where

$$\lambda_{p,j} := \sum_{i=p}^{n/2} \frac{c_{2i}^n \,(2i+j)!}{(2i-2p+j)!}.$$

Similarly,

$$\sum_{k=0}^{n} c_k^n Q_{k+1}(1) = -\sum_{k=0}^{n} c_k^n Q_{k+1}(-1) = \sum_{p=0}^{n/2} \lambda_{p,1} \epsilon^p. \tag{6}$$

The tau parameters are fixed forcing (4) to satisfy the boundary conditions of (1):

$$\tau_0 = \frac{\gamma_1 + \gamma_2}{2 \sum_{k=0}^{n} c_k^n Q_k(1)} = (\gamma_1 + \gamma_2) \left(2 + \sum_{p=1}^{n/2} \lambda_{p,0} \epsilon^p \right)^{-1},$$

$$\tau_1 = \frac{\gamma_1 - \gamma_2}{2 \sum_{k=0}^{n} c_k^n Q_{k+1}(1)} = (\gamma_1 - \gamma_2) \left(2 + \sum_{p=1}^{n/2} \lambda_{p,1} \epsilon^p \right)^{-1}.$$

We then have the following theorem.

THEOREM 1. *Let us assume that n is an even positive integer. Then,*

$$\tau_0 = \frac{\gamma_1 + \gamma_2}{2 + g_0(\epsilon; n)} \quad \text{and} \quad \tau_1 = \frac{\gamma_1 - \gamma_2}{2 + g_1(\epsilon; n)},$$

where

$$g_0(\epsilon; n) := \sum_{p=1}^{n/2} \lambda_{p,0} \epsilon^p \quad \text{and} \quad g_1(\epsilon; n) := \sum_{p=1}^{n/2} \lambda_{p,1} \epsilon^p.$$

COROLLARY 1. *Under the assumptions of Theorem 1, we have the following.*

(a) *Given $n \in \mathbf{N}$,*

$$\lim_{\epsilon \to 0} \tau_0 = \frac{\gamma_1 + \gamma_2}{2} \quad \text{and} \quad \lim_{\epsilon \to 0} \tau_1 = \frac{\gamma_1 - \gamma_2}{2}.$$

(b) *Given $0 < \epsilon \ll 1$,*

$$\lim_{n \to \infty} \tau_0 = \lim_{n \to \infty} \tau_1 = 0.$$

PROOF. Part (a) is straightforward because

$$\lim_{\epsilon \to 0} g_0 = \lim_{\epsilon \to 0} g_1 = 0.$$

Since

$$g_0 = c_n^n n! \epsilon^{n/2} + \sum_{p=1}^{n/2-1} \lambda_{p,0} \epsilon^p,$$

using Stirling's formula, it follows that

$$\lim_{n \to \infty} c_n^n n! \epsilon^{n/2} = +\infty,$$

then

$$\lim_{n \to \infty} g_0 = \lim_{n \to \infty} g_1 = +\infty,$$

which is (b).

3. UNIFORM CONVERGENCE OF TAU METHOD APPROXIMATIONS

Let $E_n := y - y_n$. Taking the difference between (1) and (3), we obtain

$$\begin{gathered} -\epsilon E_n''(x) + E_n(x) = H(x), \qquad x \in [-1,1], \\ E_n(-1) = E_n(1) = 0, \end{gathered} \tag{7}$$

with $H(x) = -(\tau_0 + \tau_1 x) V_n(x)$. E_n is given analytically by

$$E_n(x) = \frac{\alpha^2}{W} \left\{ u_2(x) \int_{-1}^{x} u_1(t) H(t)\, dt + u_1(x) \int_x^1 u_2(t) H(t)\, dt \right\},$$

where $\alpha := \epsilon^{-1/2}$, $W = 2\alpha(1 - e^{4\alpha})$, and

$$u_1(x) := e^{\alpha(2+x)} - e^{-\alpha x}, \qquad u_2(x) := e^{\alpha(2-x)} - e^{\alpha x}.$$

But

$$u_2(x) \int_{-1}^{x} u_1(t)\, dt + u_1(x) \int_x^1 u_2(t)\, dt \le \frac{2}{\alpha},$$

as u_1 and u_2 are positive functions and $x \in [-1,1]$. Therefore,

$$|E_n(x)| < \frac{e^{4\alpha}}{e^{4\alpha} - 1} |\tau_0| + |\tau_1|.$$

We then deduce the following theorem.

THEOREM 2. *Under the assumptions of Theorem 1, we have the following.*

(a) *Given* $n \in \mathbf{N}$,

$$\lim_{\epsilon \to 0} \|E_n\| \le |\tau_0| + |\tau_1|.$$

(b) *Given* $0 < \epsilon \ll 1$,

$$\lim_{n \to \infty} \|E_n\| = 0.$$

From Theorem 2, it follows that if a step-by-step Tau Method is applied to problem (1), then the estimations of the τ parameters given by Theorem 2 of El-Daou and Ortiz [6] remain valid independently of ϵ. In this case, convergence is enhanced further by segmentation, through the dependence of the Tau Method error on the length of the interval of approximation.

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